

SINGULARITIES AND BIFURCATIONS OF 3-DIMENSIONAL POISSON STRUCTURES

BY

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ABSTRACT

We give a normal form for families of 3-dimensional Poisson structures. This allows us to classify singularities with nonzero 1-jet and typical bifurcations. The Appendix contains corollaries on classification of families of integrable 1-forms on \mathbf{R}^3 .

1. Introduction and main results

After it was understood that Poisson structures play a significant role in classical and quantum mechanics, Poisson geometry became an attractive branch of mathematics. The first extensively studied Poisson structures were the regular ones ([Li]); later there appeared an interest in the study of singularities ([Ar], [We1]). The singular Poisson structures, including homogeneous ones, appear in the study of integrable Hamiltonian systems. In [Ar] such a Poisson structure on the base of versal deformation of singularities was constructed, and normal forms for singular 2-dimensional Poisson structures were given. In the 2-dimensional case the classification is similar to that of functions. This is not so beginning

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from the 3-dimensional case because of the Jacobi identity which starts to play an important role in all classification results.

This paper is devoted to the 3-dimensional case. We classify local families P_ϵ of Poisson structures on \mathbb{R}^3 such that P_0 has a singular point $0 \in \mathbb{R}^3$ (i.e., $P_0(0) = 0$) and $j_0^1 P_0$ does not vanish. This allows us to classify individual germs of Poisson structures and bifurcations which hold in generic 1-parameter families.

All objects are assumed to be of the class C^∞ ; all families are assumed to depend smoothly on parameters. Let M be a smooth manifold and N the ring of smooth functions on M . Recall that a Poisson structure on M is a composition law $(f, g) \mapsto \{f, g\}$ on N which endows N with a Lie algebra structure and satisfies the condition $\{fg, h\} = f\{g, h\} + g\{f, h\}$ for every f, g and h in N . Such a composition law is called a Poisson bracket. It is determined locally by the brackets $\{x_i, x_j\}$ where (x_1, \dots, x_n) are local coordinates. A Poisson structure can also be viewed as the 2-vector $P = \sum_{i < j} \{x_i, x_j\} \partial/\partial x_i \wedge \partial/\partial x_j$; then the Poisson bracket of two functions f and g is given by the relation $\{f, g\} = P(df, dg)$. By local equivalence of Poisson structures we always mean the equivalence with respect to the natural action of the group of local diffeomorphisms. Two local families P_ϵ and \tilde{P}_ϵ are called equivalent if there exists a family of diffeomorphisms ϕ_ϵ such that $\phi_0(0) = 0$ and $(\phi_\epsilon)_* P_\epsilon = \tilde{P}_\epsilon$ for all small ϵ . In this definition it is not required that $\phi_\epsilon(0) = 0$ as $\epsilon \neq 0$.

If M is a 3-dimensional orientable manifold then any volume form Ω induces a 1-1 correspondence between Poisson structures and integrable 1-forms. This correspondence is realized by the isomorphism $b: Z \rightarrow i_Z \Omega$ between p -vectors and $(3 - p)$ -forms. Therefore all results of this paper can be reformulated as classification results for local integrable 1-forms on \mathbb{R}^3 defined up to multiplication by a nonvanishing function; see the Appendix.

Our main result, allowing one to analyse singularities and bifurcations, is a versal unfolding of any Poisson structure having nonzero 1-jet at the origin.

THEOREM 1.1: *Let P_ϵ be a local family of Poisson structures on \mathbb{R}^3 such that $P(0) = 0$, $j_0^1 P \neq 0$, where $P = P_0$. Assume that $j_0^1 P$ is not isomorphic to the Poisson structure*

$$(1.1) \quad \{x, y\} = 0, \quad \{y, z\} = y, \quad \{z, x\} = -x.$$

Then P_ϵ is equivalent to a local family of the form

$$(1.2) \quad \{x, y\} = z, \quad \{y, z\} = U_\epsilon(x, y, z), \quad \{z, x\} = V_\epsilon(x, y, z)$$

and there exists a family of formal changes of the coordinates x, y, z , centered at the point $x = y = z = 0$ for all ϵ , reducing (1.2) to the form

$$(1.3) \quad \{x, y\} = z, \{y, z\} = \frac{\partial \hat{f}_\epsilon(x, y)}{\partial x} + z \frac{\partial \hat{g}_\epsilon(x, y)}{\partial x}, \{z, x\} = \frac{\partial \hat{f}_\epsilon(x, y)}{\partial y} + z \frac{\partial \hat{g}_\epsilon(x, y)}{\partial y},$$

where the formal series \hat{f}_ϵ and \hat{g}_ϵ satisfy the relations

$$(1.4) \quad \hat{f}_\epsilon(0) = \hat{g}_\epsilon(0) = 0, \quad d\hat{f}_\epsilon \wedge d\hat{g}_\epsilon \equiv 0.$$

This normal form is, to some extent, a generalization of the Bogdanov–Takens formal normal form for families of vector fields on a plane (see [AI]). Given a family of vector fields $v_\epsilon(x, z) = \alpha_\epsilon(x, z)\partial/\partial x + \beta_\epsilon(x, z)\partial/\partial z$ on the (x, z) plane, we can associate to it a family of Poisson structures on \mathbb{R}^3 of the form

$$(1.5) \quad \frac{\partial}{\partial y} \wedge v_\epsilon(x, z).$$

It follows from the proof of Theorem 1.1 in section 5 that, in general, such a family is reducible to the normal form (1.3), where the series $\hat{f}_\epsilon(x, y)$ and $\hat{g}_\epsilon(x, y)$ do not depend on y . Therefore (1.3) takes the form $\frac{\partial}{\partial y} \wedge (z \frac{\partial}{\partial x} + (A_\epsilon(x) + zB_\epsilon(x)) \frac{\partial}{\partial z})$ which corresponds to the Bogdanov–Takens normal form.

Using the normal form (1.3)–(1.4) we distinguish the following singularity classes. In what follows $\hat{f} = \hat{f}_0$ and $\hat{g} = \hat{g}_0$, where \hat{f}_ϵ and \hat{g}_ϵ are the functional parameters in the normal form (1.3).

1) A Poisson structure P has a V singularity at the origin if either $j_0^1 P$ is isomorphic to the Poisson structure (1.1) or $j_0^1 \hat{g} \neq 0$.

2) A Poisson structure has a $so(3)$ singularity at the origin if $j_0^2 \hat{f}$ is R -equivalent to $x^2 + y^2$ (then $j_0^1 \hat{g} = 0$ by (1.4)).

3) A Poisson structure has a $sl(2)$ singularity at the origin if $j_0^2 \hat{f}$ is R -equivalent to $x^2 - y^2$ or to $-x^2 - y^2$ (then $j_0^1 \hat{g} = 0$ by (1.4)).

4) A Poisson structure has an A^+ (resp. A^-) singularity at the origin if $j_0^2 \hat{f}$ is R -equivalent to x^2 (resp. $-x^2$) and $j_0^1 \hat{g} = 0$. An A singularity is either an A^+ or A^- singularity.

5) A Poisson structure has a N singularity at the origin if $j_0^2 \hat{f} = 0$ and $j_0^1 \hat{g} = 0$. Within N singularities we will study only N^+ and N^- singularities — the cases where $j_0^2 \hat{g}$ is R -equivalent to $x^2 + y^2$ and $x^2 - y^2$ respectively.

It is clear that any Poisson structure P such that $P(0) = 0, j_0^1 P \neq 0$ has at the origin one of these 5 types of singularities. The singularity classes $V, so(3), sl(2), A$ and N are related to the classification of 3-dimensional Lie algebras since each

of the singularity classes is distinguished by a condition on the 1-jet at the origin of a Poisson structure, and the 1-jet of any Poisson structure P , $P(0) = 0$, can be identified with a Lie algebra. It is not hard to check the following facts.

a) The $so(3)$ and $sl(2)$ singularities correspond to the Lie algebras $so(3)$ and $sl(2)$ respectively (up to isomorphism).

b) The Lie algebras corresponding to the other singularity classes are isomorphic to a Lie algebra of the form $[e_1, e_2] = 0$, $[e_1, e_3] = b_{1,1}e_1 + b_{1,2}e_2$, $[e_2, e_3] = b_{2,1}e_1 + b_{2,2}e_2$ with real parameters $b_{i,j}$.

c) Let $B = (b_{i,j})$. The V singularities are distinguished by the condition $trace B \neq 0$, the A singularities by the condition $trace B = 0$, $det B \neq 0$ ($det B > 0$ in the case of A^+ singularities and $det B < 0$ in the case of A^- singularities), and the N singularities by the condition $trace B = 0$, $det B = 0$, $B \neq 0$. The singularity classes N^+ and N^- cannot be distinguished in terms of the 1-jet of a Poisson structure (they are distinguished by a condition on $j_0^2 P$).

The singularities $so(3)$ and $sl(2)$ are well known due to the works [Co1, Co2, We1]. A Poisson structure having a $so(3)$ or $sl(2)$ singularity at the origin is formally equivalent to the linear Poisson structure

$$\{x, y\} = z, \{y, z\} = x, \{z, x\} = \pm y,$$

where the sign $+$ (resp. $-$) corresponds to the $so(3)$ (resp. $sl(2)$) singularity. In the case of $so(3)$ singularities this normal form also holds in the C^∞ category.

The $so(3)$ and $sl(2)$ singularities are isolated and irremovable: if there is a family P_ϵ , $\epsilon \in \mathbb{R}^l$, of Poisson structures such that P_0 has a $so(3)$ (resp. $sl(2)$) singularity at the origin, then there is a neighbourhood W of $0 \in \mathbb{R}^l$ and a neighbourhood U of $0 \in \mathbb{R}^3$ such that for any $\epsilon \in W$ the Poisson structure P_ϵ has a unique singular point in U at which a $so(3)$ (resp. $sl(2)$) singularity holds.

The beginning of the classification of V singularities can be found in the work [Du]. The classification reduces to the orbital classification of vector fields on a plane due to the following result: a local family P_ϵ of Poisson structures such that P_0 has a V singularity at the origin is C^∞ equivalent to a family of the form (1.5). In view of the correspondence between 3-dimensional Poisson structures and integrable 1-forms, this result is analogous to the well-known Kupka phenomenon [Ku]: the local classification of integrable 1-forms ω on \mathbb{R}^3 such that $d\omega(0) \neq 0$ reduces to the classification of arbitrary 1-forms on a plane.

It follows that the V singularities are also irremovable, but their geometry essentially differs from that for the $so(3)$ and $sl(2)$ singularities: if a Poisson structure has a V singularity at the origin then it also has a V singularity at

each point of a smooth curve passing through the origin. A detailed study of the V singularities is contained in section 2.

The A singularities are studied in section 3. These singularities can be removed by a small perturbation of an individual Poisson structure, but they are irremovable in 1-parameter families of Poisson structures. We give a normal form for any deformation of an algebraically isolated A singularity. In particular, an individual Poisson structure having an algebraically isolated A singularity at the origin is formally equivalent to a Poisson structure of the form

$$H(x, y, z) \left(z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \pm x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \pm y^m \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} \right),$$

where $m \geq 2$ and H is a nonvanishing function. This statement, expressed in terms of integrable 1-forms, can also be obtained as a corollary of the results of Moussu [Mo] or Malgrange [Ma].

The geometry of a generic 1-parameter perturbation P_ϵ of a Poisson structure P having a generic A singularity at the origin is as follows. If $\epsilon < 0$ then P_ϵ contains no singular points near the origin; the Poisson structure $P = P_0$ has an isolated singular point which decomposes into two singular points near the origin as $\epsilon > 0$. These singular points are both $sl(2)$ singularities of P_ϵ if P has an A^- singularity at the origin. If P has an A^+ singularity at the origin then one of the singular points is a $so(3)$ singularity and the other is a $sl(2)$ singularity. It is remarkable that no perturbation of an arbitrary algebraically isolated A singularity leads to a V singularity whereas the A singularities of the Lie algebras are adjoint to the V singularities.

The most difficult are the N^+ and N^- singularities studied in section 4. We prove that generic N^+ and N^- singularities are irremovable in 1-parameter families of Poisson structures. This is a bit surprising since, on the level of the Lie algebras, the N singularities are typical only in 2-parameter families. We prove that there are three types of bifurcations in generic 1-parameter families P_ϵ such that P_0 has a generic N^+ or N^- singularity at the origin:

a) If $\epsilon \leq 0$ then P_ϵ has an isolated singular point which is a $so(3)$ singularity if $\epsilon < 0$ and an N^+ singularity if $\epsilon = 0$. If $\epsilon > 0$ then the set of singular points of P_ϵ consists of an isolated singular point and a closed curve. The isolated singular point is a $sl(2)$ singularity, and each point of the curve is a V singularity.

b) If $\epsilon \leq 0$ then P_ϵ has an isolated singular point which is a $sl(2)$ singularity if $\epsilon < 0$ and an N^+ singularity if $\epsilon = 0$. If $\epsilon > 0$ then the set of singular points of P_ϵ consists of an isolated singular point and a closed curve. The isolated singular point is a $so(3)$ singularity, and each point of the curve is a V singularity.

c) The set of singular points of P_ϵ has the form (in suitable coordinates) $\{z = 0, x^2 - y^2 - \epsilon = 0\} \cup \{x = y = z = 0\}$. Any singular point except the origin is a V singularity. The origin is a $sl(2)$ singularity if $\epsilon \neq 0$ and an N^- singularity if $\epsilon = 0$.

The type of the bifurcation and the type of V singularities (node, saddle, focus) that appear can be determined in terms of the 3-jet of P_0 .

2. V -singularities

Using the **curl** of a Poisson structure we show in section 2.1 that the classification of V singularities reduces to the orbital classification of vector fields on a plane (analogous to the Kupka phenomenon [Ku]). Applying known results on the latter classification we obtain, in section 2.2, corollaries on normal forms, geometry and bifurcations.

The V singularities are never isolated — they hold at points of smooth curves. We distinguish hyperbolic V singularities (node, saddle and focus) and saddle-node V singularities. The hyperbolic V singularities are irremovable, and generic saddle-node V singularities are irremovable in 1-parameter families (in generic 1-parameter families the saddle-node bifurcation holds). The results of section 2 continue the results of the paper [Du], where the nonresonant V singularities were classified.

2.1. REDUCTION TO VECTOR FIELDS. A volume form Ω on \mathbb{R}^3 induces the isomorphism $b: Z \rightarrow i_Z\Omega$ between p -vectors and $(3-p)$ -forms on \mathbb{R}^3 . The curl of a Poisson structure P with respect to Ω is defined to be the vector field $X = b^{-1}(d(b(P)))$. It is known (see [DH], [We2]) that for any Ω the curl X is an infinitesimal symmetry of P , i.e., $[X, P] = 0$. If $P(0) = 0$ then the vector $X(0)$ does not depend on Ω . Computing the curl of the Poisson structure (1.3) with respect to the volume form $dx \wedge dy \wedge dz$, we obtain

$$X = -\frac{\partial g}{\partial y} \frac{\partial}{\partial x} + \frac{\partial g}{\partial x} \frac{\partial}{\partial y}.$$

This relation proves the following statement.

PROPOSITION 2.1: *For a Poisson structure P , $P(0) = 0$, the origin is a V singularity if and only if the curl of P with respect to some (and then any) volume form does not vanish at the origin.*

Assume that we have a family P_ϵ of Poisson structures such that P_0 has a V singularity at the origin. Let X_ϵ be the curl of P_ϵ with respect to a fixed volume

form Ω . By Proposition 2.1 there exists a coordinate system (depending smoothly on ϵ) such that $X_\epsilon = \partial/\partial y$. In the 3-dimensional case the Jacobi formula implies $X_\epsilon \wedge P_\epsilon = 0$. It follows from this relation and the relation $[X_\epsilon, P_\epsilon] = 0$ that in the chosen coordinate system P_ϵ has the form (1.5). Returning to the characterization of V singularities in terms of the 1-jet of a Poisson structure, we conclude that the sum of the eigenvalues of the vector field v_0 in (1.5) is different from zero. So, we have proved the following statement.

PROPOSITION 2.2: *Any local family P_ϵ of Poisson structures on \mathbb{R}^3 such that P_0 has a V singularity at the origin is equivalent to a family of the form*

$$\frac{\partial}{\partial y} \wedge v_\epsilon, \quad v_\epsilon = \alpha_\epsilon(x, z) \frac{\partial}{\partial x} + \beta_\epsilon(x, z) \frac{\partial}{\partial z},$$

where the sum of the eigenvalues of the linearization at the origin of the vector field v_0 is different from zero.

Consider now two families of Poisson structures

$$P_\epsilon = \frac{\partial}{\partial y} \wedge v_\epsilon \quad \text{and} \quad \tilde{P}_\epsilon = \frac{\partial}{\partial y} \wedge \tilde{v}_\epsilon,$$

where

$$v_\epsilon = \alpha_\epsilon(x, z) \frac{\partial}{\partial x} + \beta_\epsilon(x, z) \frac{\partial}{\partial z} \quad \text{and} \quad \tilde{v}_\epsilon = \tilde{\alpha}_\epsilon(x, z) \frac{\partial}{\partial x} + \tilde{\beta}_\epsilon(x, z) \frac{\partial}{\partial z}.$$

Assume that the families v_ϵ and \tilde{v}_ϵ are orbitally equivalent, i.e., there exists a family ϕ_ϵ of local diffeomorphisms of the (x, z) plane such that $(\phi_\epsilon)_* v_\epsilon = h_\epsilon \tilde{v}_\epsilon$, where h_ϵ is a family of functions such that $h_0(0) \neq 0$. Then the diffeomorphism $(x, z) \mapsto \phi_\epsilon(x, z), y \mapsto y/h_\epsilon$ brings P_ϵ to \tilde{P}_ϵ . So, the orbital equivalence of the families v_ϵ and \tilde{v}_ϵ implies the equivalence of the families P_ϵ and \tilde{P}_ϵ . The inverse statement holds under the assumption that the fields v_0 and \tilde{v}_0 have isolated singularities at the origin. In fact, let ψ_ϵ be a family of diffeomorphisms sending P_ϵ to \tilde{P}_ϵ . Since the y axis is the set of singular points for both P_0 and \tilde{P}_0 , the diffeomorphism ψ_0 sends the plane $y = 0$ to a surface transversal to the y axis. This property remains true for the diffeomorphism ψ_ϵ , when ϵ is small. Therefore there exists a family of functions $g_\epsilon(x, z)$ such that the superposition μ_ϵ of ψ_ϵ with the diffeomorphism $\nu_\epsilon: (x, y, z) \rightarrow (x, y - g_\epsilon(x, z), z)$ preserves the plane $y = 0$ for each small enough ϵ . The diffeomorphism ν_ϵ is a symmetry of the Poisson structures P_ϵ and \tilde{P}_ϵ . It follows that the superposition μ_ϵ also brings P_ϵ to \tilde{P}_ϵ . It is easy to check that the restriction of μ_ϵ to the plane $y = 0$ brings v_ϵ to \tilde{v}_ϵ multiplied by a nonvanishing function, i.e., the families v_ϵ and \tilde{v}_ϵ are orbitally equivalent. We have proved the following statement reducing the classification of V singularities to the orbital classification of vector fields.

PROPOSITION 2.3: *Let*

$$v_\epsilon = \alpha_\epsilon(x, z) \frac{\partial}{\partial x} + \beta_\epsilon(x, y) \frac{\partial}{\partial z} \quad \text{and} \quad \tilde{v}_\epsilon = \tilde{\alpha}_\epsilon(x, z) \frac{\partial}{\partial x} + \tilde{\beta}_\epsilon(x, y) \frac{\partial}{\partial z}$$

be two local families of vector fields such that the vector fields v_0 and \tilde{v}_0 have isolated singular point at the origin. The family of Poisson structures $P_\epsilon = \frac{\partial}{\partial y} \wedge v_\epsilon$ is equivalent to the family $\tilde{P}_\epsilon = \frac{\partial}{\partial y} \wedge \tilde{v}_\epsilon$ if and only if the family v_ϵ is orbitally equivalent to the family \tilde{v}_ϵ .

2.2. NORMAL FORMS AND GEOMETRY OF V SINGULARITIES. The results of this section are direct corollaries of Propositions 2.2 and 2.3 and results on the orbital classification of vector fields on a plane; see [AI, Bo]. At first we distinguish node, saddle, focus and saddle-node V singularities. Assume that a Poisson structure P has a V singularity at the origin. By Proposition 2.2, P is equivalent to a Poisson structure of the form $\frac{\partial}{\partial y} \wedge v$, where v is a vector field on the (x, z) plane, $v(0) = 0$. Let λ_1 and λ_2 be the eigenvalues of the linearization of the vector field v in this normal form. By Proposition 2.3 they are invariantly related to P up to multiplication by a common factor, and we will say that λ_1 and λ_2 are the eigenvalues of P . Note that $\lambda_1 + \lambda_2 \neq 0$; see Proposition 2.2.

We will say that a V singularity is hyperbolic if $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Since $\lambda_1 + \lambda_2 \neq 0$ this means that the spectrum of the linearization of the vector field v does not intersect the imaginary axis.

Within hyperbolic V singularities we distinguish node V singularities corresponding to the case where λ_1 and λ_2 are real nonzero numbers of the same sign, saddle V singularities corresponding to the case where λ_1 and λ_2 are real nonzero numbers of different signs, and focus V singularities corresponding to the case where $\lambda_{1,2} = a \pm ib$, $a \neq 0$, $b \neq 0$.

If a V singularity is not hyperbolic then one of the eigenvalues is equal to zero and the second is different from zero. In this case we will say that the origin is a saddle-node V singularity. Within saddle-node V singularities there exists a degeneration of infinite codimension corresponding to the case where the origin is not an algebraically isolated singular point of the vector field v (this is impossible for hyperbolic V singularities). If such a degeneration holds we will say that the origin is an exclusive saddle-node V singularity.

THEOREM 2.4: *Assume that P has a hyperbolic or nonexclusive saddle-node V singularity at the origin. Then the set of singular points of P in a small enough neighbourhood of the origin is a smooth curve γ . The germs of P at points of γ are C^∞ equivalent to the germ of P at the origin and C^∞ equivalent to one of*

the following normal forms:

$$(2.1) \quad \{x, z\} = 0, \quad \{x, y\} = z, \quad \{z, y\} = \theta x + z,$$

$$\theta \in \mathbb{R} - \{0\};$$

$$(2.2) \quad \{x, z\} = 0, \quad \{x, y\} = Nx + \delta z^N, \quad \{z, y\} = z,$$

$$N \in \{1, 2, 3, 4, \dots\}, \quad \delta \in \{0, 1\};$$

$$(2.3) \quad \{x, z\} = 0, \quad \{x, y\} = -\frac{p}{q}x + \delta x^{q+1}z^p + \delta a x^{2q+1}z^{2p}, \quad \{z, y\} = z,$$

$$\delta \in \{1, -1, 0\}, a \in \mathbb{R};$$

$$(2.4) \quad \{x, z\} = 0, \quad \{x, y\} = \delta^{p+1}x^{p+1} + ax^{2p+1}, \quad \{z, y\} = z,$$

$$p \geq 1, \quad \delta \in \{1, -1\}, \quad a \in \mathbb{R}.$$

The normal form (2.1) holds for focus V singularities and nonresonant node or saddle V singularities, i.e., for node V singularities such that neither λ_1/λ_2 nor λ_2/λ_1 is an integer number and for saddle V singularities such that λ_1/λ_2 is not a rational number. Here λ_1 and λ_2 are the eigenvalues of P . The normal form (2.2) holds for resonant node V singularities, the normal form (2.3) holds for resonant saddle V singularities (in this normal form p and q are positive integer numbers and p/q is an irreducible fraction), and the normal form (2.4) holds for nonexclusive saddle-node V singularities.

Since the hyperbolic singular points of vector fields are irremovable, the hyperbolic V singularities are irremovable under a small perturbation of a Poisson structure. Namely, if P has a node (resp. saddle, focus) V singularity at the origin and P_ϵ is a family of Poisson structures, $\epsilon \in \mathbb{R}^l$, such that $P_0 = P$, then there exist a neighbourhood of the origin $U \subset \mathbb{R}^3$ and a neighbourhood of the origin $W \subset \mathbb{R}^l$ such that for any $\epsilon \in W$ the set of singular points of P_ϵ in U is a smooth curve γ_ϵ . The family γ_ϵ depends smoothly on ϵ and each point of γ_ϵ is also a node (resp. saddle, focus) V singularity of P_ϵ .

The saddle-node V singularities are irremovable only in 1-parameter families of Poisson structures. Using the correspondence between V singularities and vector fields on a plane, we will say that a saddle-node V singularity is generic if $p = 1$ in the normal form (2.4). If P has a generic saddle-node singularity at the origin

and P_ϵ is a 1-parameter deformation of P , then in a suitable coordinate system (depending smoothly on ϵ) the 2-jet of the family P_ϵ has the form $\{x, z\} = 0$, $\{x, y\} = f_0(\epsilon) + f_1(\epsilon)x + x^2$, $\{z, y\} = z$. We will say that P_ϵ is a generic deformation of P if $f'_0(0) \neq 0$. In this case the following (up to the change $\epsilon \rightarrow -\epsilon$) saddle-node bifurcation holds (analogous to the well known saddle-node bifurcation for vector fields). There exists a neighbourhood of the origin $U \subset \mathbb{R}^3$ and a neighbourhood of the origin $W \subset \mathbb{R}$ such that if $\epsilon \in W$ is a negative number then U contains no singular points of P_ϵ and if $\epsilon \in W$ is a positive number then the set of singular points of P_ϵ in U consists of two disjoint smooth curves. The points of the first curve are saddle V singularities and the points of the second curve are node V singularities of P_ϵ .

3. A singularities

This section contains results on algebraically isolated A singularities. Recall that a Poisson structure $\{x, y\} = B(x, y, z)$, $\{y, z\} = C(x, y, z)$, $\{z, x\} = D(x, y, z)$ has an algebraically isolated singularity at the origin if the factor ring of the ring of all formal series over the ideal generated by the formal series of the functions B, C and D has finite dimension over \mathbb{R} . It is clear that an A singularity is algebraically isolated if and only if the formal series $\hat{f}_0(x, y)$ in the normal form (1.3) is R -equivalent to $\pm x^2 \pm y^{m+1}$ for some $m \geq 2$. It follows that all A singularities, except degenerations of infinite codimension, are algebraically isolated. We will say that an A singularity is generic if $m = 2$.

THEOREM 3.1: *Let P_ϵ be a local family of Poisson structures of the form (1.2) such that P_0 has an algebraically isolated A singularity at the origin p_0 . There is a family of smooth changes of coordinates parametrized by ϵ such that in the new coordinate system P_ϵ has the form*

$$(3.1) \quad H_\epsilon(x, y, z) \left(z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \pm x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + (\delta y^m + \sum_{i=0}^{m-2} h_i(\epsilon) y^i) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} \right)$$

modulo a flat field of 2-vectors at the point p_0 , where $m \geq 2$, $H_0(0) > 0$, $h_0(0) = \dots = h_{m-2}(0) = 0$, $\delta = \pm 1$ if m is odd and $\delta = 1$ if m is even.

The sign + (resp. -) in (3.1) corresponds to the case of A^+ (resp. A^-) singularities. Note that in general the point p_0 is the origin of the coordinate system of the normal form (3.1) only if $\epsilon = 0$.

In particular, any 1-parameter unfolding of a generic A singularity reduces to the formal normal form

$$(3.2) \quad H_\epsilon(x, y, z) \left(z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \pm x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + (y^2 + h_0(\epsilon)) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} \right).$$

A 1-parameter unfolding of a generic A singularity will be called generic if the function h_0 in this normal form satisfies the condition $h'_0(0) \neq 0$. The following theorem says that A singularities are irremovable in 1-parameter families of Poisson structures and a generic individual A singularity decomposes, under a small perturbation, into two singular points; the type of singularity at these points depends on the type of singularity of P at the origin (A^+ or A^-).

THEOREM 3.2: *Let P_ϵ be a generic 1-parameter unfolding of a generic Poisson structure P having an A singularity at the origin. Then the following bifurcation (up to the change $\epsilon \rightarrow -\epsilon$) holds in a neighbourhood of the origin $U \subset \mathbb{R}^3$ and in a neighbourhood of the origin $W \subset \mathbb{R}$. If $\epsilon \in W$ and $\epsilon < 0$ then U contains no singular points of P_ϵ ; if $\epsilon \in W$ and $\epsilon > 0$ then U contains two singular points of P_ϵ . If the singularity of P has the type A^+ then one of these singular points is a $so(3)$ singularity and the other is a $sl(2)$ singularity. If the singularity of P has the type A^- then the two singular points are $sl(2)$ singularities.*

Theorem 3.2 is an easy corollary of the normal form (3.2). One also can obtain, using the normal form (3.1), that any (not necessarily generic) algebraically isolated A^- singularity decomposes, under a small perturbation, into $sl(2)$ singularities, and any algebraically isolated A^+ singularity decomposes into $so(3)$ and $sl(2)$ singularities. It follows that there is no adjacency between algebraically isolated A singularities and V singularities. Note that such an adjacency holds for Lie algebras — a suitable perturbation of any Lie algebra of the type A (in the class of Lie algebras) leads to a Lie algebra of the type V .

Proof of Theorem 3.1: We will use Theorem 1.1, i.e., the formal normal form (1.3)–(1.4). Since P_0 has an algebraically isolated singularity at the origin then the 1-form $df_0(x, y)$ also has an algebraically isolated singularity at the origin of the (x, y) plane. Using this observation and results of the paper [Mo] we conclude that (1.4) implies that $\hat{g}_\epsilon = \hat{\lambda}_\epsilon \circ \hat{f}_\epsilon$, where $\hat{\lambda}_\epsilon$ is a family of formal series in one variable depending smoothly on ϵ .

Now we will show that P_ϵ admits a family of Casimir functions, i.e., there exists a family $\hat{C}_\epsilon = \hat{C}_\epsilon(x, y, z)$ of nonzero formal series such that $\hat{C}_\epsilon(0) = 0$ and $\{\hat{C}_\epsilon, x\} = \{\hat{C}_\epsilon, y\} = \{\hat{C}_\epsilon, z\} = 0$. The latter relation means that $P_\epsilon(d\hat{C}_\epsilon, d\hat{h}) = 0$ for any formal series \hat{h} .

Taking into account the relation $\hat{g}_\epsilon = \hat{\lambda}_\epsilon \circ \hat{f}_\epsilon$ it is natural to seek a family of Casimir functions in the form $\hat{C}_\epsilon(x, y, z) = \hat{G}_\epsilon(z, \hat{f}_\epsilon)$, where $\hat{G}_\epsilon = \hat{G}_\epsilon(z, w)$ is a family of formal series in two variables. A simple computation shows that

$$P_\epsilon(d\hat{C}_\epsilon, dz) = 0, \quad P_\epsilon(d\hat{C}_\epsilon, dx) = \hat{Q}_\epsilon \frac{\partial \hat{f}_\epsilon}{\partial y}, \quad P_\epsilon(d\hat{C}_\epsilon, dy) = -\hat{Q}_\epsilon \frac{\partial \hat{f}_\epsilon}{\partial x},$$

where

$$\hat{Q}_\epsilon = (1 + z\hat{\lambda}'_\epsilon(\hat{f}_\epsilon))\frac{\partial \hat{G}_\epsilon}{\partial z}(z, \hat{f}_\epsilon) - z\frac{\partial \hat{G}_\epsilon}{\partial w}(z, \hat{f}_\epsilon).$$

The equation $\hat{Q}_\epsilon = 0$ has a formal solution $\hat{G}_\epsilon(z, w)$ of the form

$$\hat{G}_\epsilon(z, w) = w + z^2/2 + a(\epsilon)w^2 + \hat{R}_\epsilon(z, w),$$

where \hat{R}_ϵ is a formal series with zero 2-jet for each fixed ϵ . Therefore P_ϵ admits a family of Casimir functions $\hat{C}_\epsilon(x, y, z)$ such that

$$\hat{C}_0(x, y, z) = \hat{f}_0(x, y) + z^2/2 + a(0)\hat{f}_0^2(x, y) + R_0(z, \hat{f}_0(x, y)).$$

The series $\hat{f}_0(x, y)$ is R -equivalent to $\pm \frac{x^2}{2} + \delta \frac{y^{m+1}}{m+1}$, where $m \geq 2$, $\delta = 1$ if m is even and $\delta \in \{-1, 1\}$ if m is odd. It follows that the series $\hat{C}_0(x, y, z)$ is R -equivalent to $\frac{z^2}{2} \pm \frac{x^2}{2} + \delta \frac{y^{m+1}}{m+1}$. Let $C_\epsilon(x, y, z)$ be a family of smooth functions such that the formal series of $C_\epsilon(x, y, z)$ is equal to $\hat{C}_\epsilon(x, y, z)$. Then there exists a new smooth system of coordinates $\tilde{x} = \tilde{x}_\epsilon, \tilde{y} = \tilde{y}_\epsilon, \tilde{z} = \tilde{z}_\epsilon$ such that

$$C_\epsilon(x, y, z) = \tilde{C}_\epsilon(\tilde{x}, \tilde{y}, \tilde{z}) = \alpha(\epsilon) + \frac{\tilde{z}^2}{2} \pm \frac{\tilde{x}^2}{2} + \delta \frac{\tilde{y}^{m+1}}{m+1} + \sum_{i=1}^{m-1} h_i(\epsilon)\tilde{y}^i,$$

where $h_i(0) = 0$; see [AVG]. Now Theorem 3.1 easily follows from the observation that the functions

$$\{\tilde{C}_\epsilon(\tilde{x}, \tilde{y}, \tilde{z}), \tilde{x}\}, \quad \{\tilde{C}_\epsilon(\tilde{x}, \tilde{y}, \tilde{z}), \tilde{y}\}, \quad \{\tilde{C}_\epsilon(\tilde{x}, \tilde{y}, \tilde{z}), \tilde{z}\}$$

are flat at the point $x = y = z = 0$. ■

4. N^+ and N^- singularities

In this section we show that the N^+ and N^- singularities are typical in 1-parameter families of Poisson structures (though they are typical only in 2-parameter families of Lie algebras) and analyse typical bifurcations in 1-parameter families P_ϵ such that P_0 has an N^+ or N^- singularity at the origin. The analysis of bifurcations is based on the following proposition.

PROPOSITION 4.1: *Any local family of Poisson structures P_ϵ such that P_0 has an N^+ or N^- singularity at the origin is equivalent to a family of the form*

$$\{x, y\} = z, \quad \{y, z\} = A_\epsilon(x, y)(z + C_\epsilon(x, y)) + z^2 Q_{1,\epsilon}(x, y, z),$$

$$(4.1) \quad \{z, x\} = B_\epsilon(x, y)(z + C_\epsilon(x, y)) + z^2 Q_{2,\epsilon}(x, y, z),$$

where the formal series of the functions A_ϵ , B_ϵ and C_ϵ have the form

$$(4.2) \quad \begin{aligned} \hat{A}_\epsilon(x, y) &= x (\lambda_0(\epsilon) + \lambda_1(\epsilon)(x^2 \pm y^2) + \lambda_2(\epsilon)(x^2 \pm y^2)^2 + \dots), \\ \hat{B}_\epsilon(x, y) &= \pm y (\lambda_0(\epsilon) + \lambda_1(\epsilon)(x^2 \pm y^2) + \lambda_2(\epsilon)(x^2 \pm y^2)^2 + \dots), \\ \hat{C}_\epsilon(x, y) &= \mu_0(\epsilon) + \mu_1(\epsilon)(x^2 \pm y^2) + \mu_2(\epsilon)(x^2 \pm y^2)^2 + \dots, \end{aligned}$$

and the functions $\lambda_0(\epsilon)$ and $\mu_0(\epsilon)$ satisfy the conditions

$$(4.3) \quad \lambda_0(0) \neq 0, \quad \mu_0(0) = 0.$$

The signs + (resp. -) in (4.2) correspond to the case where P_0 has a N^+ (resp. N^-) singularity at the origin.

This proposition will be proved at the end of the section. In terms of the normal form (4.1)–(4.2) we define a generic N^+ or N^- singularity and its generic 1-parameter unfolding. An N^+ or N^- singularity will be called generic if $\mu_1(0) \neq 0$, and a 1-parameter unfolding will be called generic if $\mu'_0(0)\mu_1(0) \neq 0$.

The bifurcations in generic 1-parameter unfoldings of generic N^+ and N^- singularities depend on the signs of the numbers

$$\kappa_1 = \lambda_0(0)\mu_1(0), \quad \kappa_2 = \lambda_0^2(0) - 8\kappa_1.$$

The main result of this section is the following theorem.

THEOREM 4.2: *Let P be a Poisson structure with a generic N^+ or N^- singularity at the origin, and let P_ϵ be a generic 1-parameter unfolding of P . There exists a coordinate system $\tilde{x} = \tilde{x}_\epsilon$, $\tilde{y} = \tilde{y}_\epsilon$, $\tilde{z} = \tilde{z}_\epsilon$, $\tilde{\epsilon} = \epsilon Q(\epsilon)$, $Q(0) \neq 0$, such that the following statements hold (locally near the point $\tilde{x} = \tilde{y} = \tilde{z} = \tilde{\epsilon} = 0$).*

1) *The set of singular points of P_ϵ has the form*

$$\{\tilde{z} = 0, \tilde{x}^2 + \tilde{y}^2 - \tilde{\epsilon} = 0\} \cup \{\tilde{x} = \tilde{y} = \tilde{z} = 0\}$$

in the case of an N^+ singularity and

$$\{\tilde{z} = 0, \tilde{x}^2 - \tilde{y}^2 - \tilde{\epsilon} = 0\} \cup \{\tilde{x} = \tilde{y} = \tilde{z} = 0\}$$

in the case of an N^- singularity.

2) The Poisson structure P_ϵ has a V singularity at any singular point except the point $\tilde{x} = \tilde{y} = \tilde{z} = 0$. Any V singularity is a node V singularity if $\kappa_2 \geq 0$ and $\kappa_1 > 0$, a saddle V singularity if $\kappa_2 \geq 0$ and $\kappa_1 < 0$, and a focus V singularity if $\kappa_2 < 0$.

3) If P has an N^+ singularity at the origin and $\kappa_1 > 0$ (resp. $\kappa_1 < 0$) then the point $\tilde{x} = \tilde{y} = \tilde{z} = 0$ is a $so(3)$ (resp. $sl(2)$) singularity of P_ϵ if $\epsilon > 0$ and a $sl(2)$ (resp. $so(3)$) singularity if $\epsilon < 0$.

4) If P has an N^- singularity at the origin then for any $\epsilon \neq 0$ the Poisson structure P_ϵ has a $sl(2)$ singularity at the point $\tilde{x} = \tilde{y} = \tilde{z} = 0$.

This theorem implies that there are three types of typical bifurcations:

a) $\epsilon < 0$: a unique singular point — a $so(3)$ singularity; $\epsilon = 0$: a unique singular point — an N^+ singularity such that $\kappa_1 > 0$; $\epsilon > 0$: a $sl(2)$ singularity at an isolated singular point and a closed curve of V singularities which are either nodes (if $\kappa_2 \geq 0$) or focuses (if $\kappa_2 < 0$).

b) $\epsilon < 0$: a unique singular point — a $sl(2)$ singularity; $\epsilon = 0$: a unique singular point — an N^+ singularity such that $\kappa_1 < 0$; $\epsilon > 0$: a $so(3)$ singularity at an isolated singular point and a closed curve of saddle V singularities.

c) The set of singular points consists of a point p and two curves given by the equation $x^2 - y^2 = \epsilon$, $z = 0$ in a coordinate system centered at the point p . The point p is a $sl(2)$ singularity if $\epsilon \neq 0$ and an N^- singularity if $\epsilon = 0$. All other singular points are node or saddle or focus V singularities depending on the signs of the numbers κ_1 and κ_2 only.

Proof of Theorem 4.2: We will restrict ourselves to the case of N^+ singularities such that $\kappa_1 > 0$ (the proof for the other cases is similar). Throughout the proof we use the normal form (4.1)–(4.2). We will assume that $\mu'_0(0)\mu_1(0) < 0$ (for if not one can change ϵ by $\tilde{\epsilon} = -\epsilon$). The first statement of Theorem 4.2 obviously follows from the normal form, and the second and third statements are corollaries of the following calculations.

a) Assume that $\epsilon < 0$. The equation $C_\epsilon(x, y) = 0$ has no solutions near the origin, therefore the point $x = y = z = 0$ is the only singular point of P_ϵ . The linearization of P_ϵ at this point has the form

$$(4.4) \quad \{x, y\} = z, \quad \{y, z\} = \mu_0(\epsilon)\lambda_0(\epsilon)x, \quad \{z, x\} = \mu_0(\epsilon)\lambda_0(\epsilon)y.$$

The relations $\epsilon < 0$, $\mu'_0(0)\mu_1(0) < 0$ and $\mu_1(0)\lambda_0(0) > 0$ imply that $\mu_0(\epsilon)\lambda_0(\epsilon) > 0$, and it follows that P_ϵ has a $so(3)$ singularity at the point $x = y = z = 0$.

b) Assume that $\epsilon > 0$. The set of singular points of P_ϵ consists of the point $x = y = z = 0$ and the closed curve Γ_ϵ given by the equations $z = 0$, $C_\epsilon(x, y) = 0$.

The linearization of P_ϵ at the point $x = y = z = 0$ has the form (4.4), but now $\epsilon > 0$ and $\mu_0(\epsilon)\lambda_0(\epsilon) < 0$. Therefore P_ϵ has a $sl(2)$ singularity at the point $x = y = z = 0$.

These calculations prove the third statement of Theorem 4.2. To prove the second statement, take a point p of the curve Γ_ϵ . The linearization of P_ϵ at p has the form

$$\{x, y\} = z, \quad \{y, z\} = A_\epsilon(p) \left(\frac{\partial C_\epsilon}{\partial x}(p)x + \frac{\partial C_\epsilon}{\partial y}(p)y + z \right),$$

$$(4.5) \quad \{z, x\} = B_\epsilon(p) \left(\frac{\partial C_\epsilon}{\partial x}(p)x + \frac{\partial C_\epsilon}{\partial y}(p)y + z \right).$$

The Jacobi identity implies

$$(4.6) \quad A_\epsilon(p) \frac{\partial C_\epsilon}{\partial y}(p) = B_\epsilon(p) \frac{\partial C_\epsilon}{\partial x}(p).$$

Assume that $B_\epsilon(p) \neq 0$. Let $u = A_\epsilon(p)x + B_\epsilon(p)y$. Taking into account (4.6) we obtain that in the coordinate system x, z, u the Poisson structure (4.5) takes the form

$$(4.7) \quad \{u, z\} = 0, \quad \{x, u\} = B_\epsilon(p)z, \quad \{x, z\} = -B_\epsilon(p)z - \frac{\partial C_\epsilon}{\partial y}(p)u.$$

Now we see that the condition $B_\epsilon(p) \neq 0$ implies that P_ϵ has a V singularity at p . On the other hand, the same conclusion holds if $A_\epsilon(p) \neq 0$. In this case we use the coordinate system y, z, u in which (4.5) takes the form

$$\{u, z\} = 0, \quad \{y, u\} = -A_\epsilon(p)z, \quad \{y, z\} = A_\epsilon(p)z + \frac{\partial C_\epsilon}{\partial x}(p)u.$$

Note now that for any point $p \in \Gamma_\epsilon$ either $A_\epsilon(p) \neq 0$ or $B_\epsilon(p) \neq 0$. Therefore P_ϵ has a V singularity at any point of the curve Γ_ϵ .

The type of V singularity (saddle, node, focus) is the same for all points of Γ_ϵ (see section 2), therefore for its determination it suffices to analyse the linear approximation at a single point of Γ_ϵ , for example, at a point q which is the intersection of Γ_ϵ with the semiaxis $\{x = z = 0, y > 0\}$. The point q has the coordinates $(0, r\sqrt{\epsilon} + o(\sqrt{\epsilon}), 0)$, where

$$r = \sqrt{-\frac{\mu'_0(0)}{\mu_1(0)}}.$$

The function B_ϵ does not vanish at q , therefore the linear approximation of P_ϵ at q is isomorphic to (4.7). Note that

$$B_\epsilon(q) = r\lambda_0(0)\sqrt{\epsilon} + o(\sqrt{\epsilon}),$$

$$\frac{\partial C_\epsilon}{\partial y}(q) = 2r\mu_1(0)\sqrt{\epsilon} + o(\sqrt{\epsilon}).$$

These relations allow us to write (4.7) in the form $r\sqrt{\epsilon}\frac{\partial}{\partial x} \wedge v_0 + o(\sqrt{\epsilon})$, where

$$v_0 = \lambda_0(0)z\frac{\partial}{\partial u} - (\lambda_0(0)z + 2\mu_1(0)u)\frac{\partial}{\partial z}.$$

It follows that the type of the V singularity of P_ϵ at q is determined by the type of singular point of the vector field v_0 . Computing the eigenvalues of v_0 we obtain the second statement of Theorem 4.2. ■

Proof of Proposition 4.1: The idea of the proof is as follows. At first we prove that in a suitable smooth coordinate system the curl of P_ϵ vanishes at the origin for all ϵ . After this we reduce P_ϵ to the formal normal form (1.3)–(1.4) and simplify this normal form using the condition on the curl. This simplification allows us to obtain the smooth normal form (4.1)–(4.2).

Take a coordinate system x, y, z (depending on ϵ) such that P_ϵ has the form

$$\{x, y\} = z, \quad \{y, z\} = c_\epsilon(x, y) + za_\epsilon(x, y) + z^2Q_{1,\epsilon}(x, y, z),$$

$$(4.8) \quad \{z, x\} = d_\epsilon(x, y) + zb_\epsilon(x, y) + z^2Q_{2,\epsilon}(x, y, z).$$

The Jacobi identity restricted to the plane $z = 0$ gives the relation

$$(4.9) \quad c_\epsilon(x, y)b_\epsilon(x, y) = d_\epsilon(x, y)a_\epsilon(x, y).$$

It is easy to see that the condition that P_0 has an N^+ or N^- singularity at the origin implies that the function $a_0(x, y)$ and $b_0(x, y)$ are differentially independent. Therefore the relation (4.9) implies the existence of a family $s_\epsilon(x, y)$ such that

$$(4.10) \quad c_\epsilon(x, y) = s_\epsilon(x, y)a_\epsilon(x, y), \quad d_\epsilon(x, y) = s_\epsilon(x, y)b_\epsilon(x, y).$$

Consider the family p_ϵ of points with coordinates x, y, z such that

$$z = 0, \quad a_\epsilon(x, y) = 0, \quad b_\epsilon(x, y) = 0.$$

It is clear that p_ϵ depends smoothly on ϵ and that p_ϵ is a singular point of P_ϵ . The linearization of P_ϵ at the point p_ϵ has the form

$$(4.11) \quad \{x, y\} = z, \quad \{y, z\} = s_\epsilon(p_\epsilon)(e_1x + e_2y), \quad \{z, x\} = s_\epsilon(p_\epsilon)(e_3x + e_4y).$$

It is easy to check that the Jacoby identity implies that any linear Poisson structure of the form (4.11) has the curl vanishing at the point $x = y = z = 0$. Therefore the curl of P_ϵ vanishes at any singular point p_ϵ .

There is no loss of generality to assume that $p_\epsilon = 0$ for all ϵ , i.e., that the curl of P_ϵ vanishes at the origin, and it what follows we assume that this condition holds. By Theorem 1.1 the family P_ϵ can be reduced, by a formal change of coordinates, to the form (1.3)–(1.4). It is easy to see that any formal change of coordinates of the form

$$(4.12) \quad \phi_\epsilon: (x, y) \rightarrow (\phi_{1,\epsilon}(x, y), \phi_{2,\epsilon}(x, y))$$

preserving the volume form $dx \wedge dy$ also preserves the normal form (1.3)–(1.4) up to the change $\hat{f}_\epsilon \rightarrow \hat{f}_\epsilon \circ \phi_\epsilon, \hat{g}_\epsilon \rightarrow \hat{g}_\epsilon \circ \phi_\epsilon$. The fact that the curl of P_ϵ vanishes at the origin implies that $d\hat{g}_\epsilon(0) = 0$ for all ϵ ; see section 2. Then the 2-jet of \hat{g}_ϵ is R -equivalent to $(x^2 \pm y^2)$, and there is a formal change of coordinates of the form (4.12) preserving the volume form $dx \wedge dy$ and reducing \hat{g}_ϵ to the form

$$(4.13) \quad \lambda_1(\epsilon)(x^2 \pm y^2) + \lambda_2(\epsilon)(x^2 \pm y^2)^2 + \dots$$

By the condition (1.4) this change of coordinates brings the series \hat{f}_ϵ to the form

$$(4.14) \quad \mu_1(\epsilon)(x^2 \pm y^2) + \mu_2(\epsilon)(x^2 \pm y^2)^2 + \dots,$$

where $\mu_1(0) = 0$ (the latter follows from the definiton of N singularities). So, P_ϵ reduces to the formal normal form (1.3), where the formal series \hat{g}_ϵ and \hat{f}_ϵ have the form (4.13) and (4.14). Now we can return to the smooth normal form (4.8), where the formal series of $c_\epsilon, d_\epsilon, a_\epsilon$ and b_ϵ are equal, respectively, to $\partial\hat{f}_\epsilon/\partial x, \partial\hat{f}_\epsilon/\partial y, \partial\hat{g}_\epsilon/\partial x, \partial\hat{g}_\epsilon/\partial y$. Proposition 4.1 is a direct corollary of this normal form and the relation (4.10). ■

5. Proof of Theorem 1.1

It is easy to see that the condition that j_0^1P is not isomorphic to (1.1) implies that there are three differentially independent functions a, b and c , vanishing at the origin, such that $P(da, db) = c$. Then $P_\epsilon(da, db) = c_\epsilon$, where c_ϵ is a family of

functions such that $c_0 = c$, and in the coordinate system $x = a, y = b, z = c_\epsilon$ the family P_ϵ has the form (1.2).

In what follows we will use the following notation. By $a_\epsilon^{(i)}(x, y, z), b_\epsilon^{(i)}(x, y, z), \dots$ we denote functions which are polynomials homogeneous of degree i , with respect to the first two coordinates, with coefficients being smooth functions of ϵ and the last coordinate. Also, we will say that a function $h_\epsilon(x, y, z)$ is affine with respect to z if it can be written in the form $h_{0,\epsilon}(x, y) + zh_{1,\epsilon}(x, y)$. Let us show that the normal form (1.3)-(1.4) is a corollary of the following lemma.

LEMMA 5.1: *Let $q \geq 0$ and P_ϵ be a family of Poisson structures of the form*

$$\{x, y\} = z + c_\epsilon^{(q+2)}(x, y, z) + c_\epsilon^{(q+3)}(x, y, z) + \dots,$$

$$(5.1) \quad \{y, z\} = a_\epsilon^{(0)}(x, y, z) + \dots + a_\epsilon^{(q-1)}(x, y, z) + a_\epsilon^{(q)}(x, y, z) + \dots,$$

$$\{z, x\} = b_\epsilon^{(0)}(x, y, z) + \dots + b_\epsilon^{(q-1)}(x, y, z) + b_\epsilon^{(q)}(x, y, z) + \dots,$$

where the functions $a_\epsilon^{(0)}, \dots, a_\epsilon^{(q-1)}$ and $b_\epsilon^{(0)}, \dots, b_\epsilon^{(q-1)}$ are affine with respect to z . There exists a change of coordinates of the form

$$X = x + \mu_\epsilon^{(q+2)}(x, y, z), \quad Y = y, \quad Z = z + \gamma_\epsilon^{(q+1)}(x, y, z) + \gamma_\epsilon^{(q+2)}(x, y, z)$$

reducing P_ϵ to the form

$$\{X, Y\} = Z + C_\epsilon^{(q+3)}(X, Y, Z) + C_\epsilon^{(q+4)}(X, Y, Z) + \dots,$$

$$(5.2) \quad \{Y, Z\} = a_\epsilon^{(0)}(X, Y, Z) + \dots + a_\epsilon^{(q-1)}(X, Y, Z) + A_\epsilon^{(q)}(X, Y, Z) + \dots,$$

$$\{Z, X\} = b_\epsilon^{(0)}(X, Y, Z) + \dots + b_\epsilon^{(q-1)}(X, Y, Z) + B_\epsilon^{(q)}(X, Y, Z) + \dots,$$

where the functions $A_\epsilon^{(q)}$ and $B_\epsilon^{(q)}$ are affine with respect to Z .

This lemma implies the existence of a family of formal change of coordinates (centered at the origin for all ϵ) reducing any Poisson structure of the form (1.2) to the form

$$\{x, y\} = z, \quad \{y, z\} = a_\epsilon(x, y) + zb_\epsilon(x, y), \quad \{y, z\} = c_\epsilon(x, y) + zd_\epsilon(x, y),$$

with some formal series $a_\epsilon, b_\epsilon, c_\epsilon$ and d_ϵ . The Jacobi identity gives the relations

$$a_\epsilon d_\epsilon = b_\epsilon d_\epsilon, \quad \frac{\partial a_\epsilon}{\partial x} = \frac{\partial c_\epsilon}{\partial y}, \quad \frac{\partial b_\epsilon}{\partial x} = \frac{\partial d_\epsilon}{\partial y},$$

and the normal form (1.3)–(1.4) follows.

Proof of Lemma 5.1: Make a change of coordinates of the form

$$(5.3) \quad \tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{z} = z(1 + e_\epsilon^{(q+1)}(x, y, z)).$$

This change transforms (5.1) to the form

$$\{\tilde{x}, \tilde{y}\} = \tilde{z} + z\tilde{c}_\epsilon^{(q+1)}(\tilde{x}, \tilde{y}, \tilde{z}) + \tilde{c}_\epsilon^{(q+2)}(\tilde{x}, \tilde{y}, \tilde{z}) + \dots,$$

$$(5.4) \quad \{\tilde{y}, \tilde{z}\} = a_\epsilon^{(0)}(\tilde{x}, \tilde{y}, \tilde{z}) + a_\epsilon^{(1)}(\tilde{x}, \tilde{y}, \tilde{z}) + \dots + a_\epsilon^{(q-1)}(\tilde{x}, \tilde{y}, \tilde{z}) + A_\epsilon^{(q)}(\tilde{x}, \tilde{y}, \tilde{z}) + \dots,$$

$$\{\tilde{z}, \tilde{x}\} = b_\epsilon^{(0)}(\tilde{x}, \tilde{y}, \tilde{z}) + b_\epsilon^{(1)}(\tilde{x}, \tilde{y}, \tilde{z}) + \dots + b_\epsilon^{(q-1)}(\tilde{x}, \tilde{y}, \tilde{z}) + B_\epsilon^{(q)}(\tilde{x}, \tilde{y}, \tilde{z}) + \dots,$$

where

$$(5.5) \quad A_\epsilon^{(q)} = a_\epsilon^{(q)} - z^2 \frac{\partial e_\epsilon^{(q+1)}}{\partial x}, \quad B_\epsilon^{(q)} = b_\epsilon^{(q)} - z^2 \frac{\partial e_\epsilon^{(q+1)}}{\partial y}.$$

Let

$$A_\epsilon^{(q)} = \alpha_{0,\epsilon}(x, y) + z\alpha_{1,\epsilon}(x, y) + z^2\alpha_{2,\epsilon}(x, y, z),$$

$$B_\epsilon^{(q)} = \beta_{0,\epsilon}(x, y) + z\beta_{1,\epsilon}(x, y) + z^2\beta_{2,\epsilon}(x, y, z).$$

Write the Jacobi identity for the Poisson structure (5.1) in the form

$$E_{0,\epsilon}(x, y) + zE_{1,\epsilon}(x, y) + z^2E_{2,\epsilon}(x, y) + z^3E_{3,\epsilon}(x, y, z) = 0.$$

Using the condition that the functions $a_\epsilon^{(0)}, \dots, a_\epsilon^{(q-1)}$ and $b_\epsilon^{(0)}, \dots, b_\epsilon^{(q-1)}$ are affine with respect to z , we obtain

$$E_{3,\epsilon}(x, y, z) = E_{3,\epsilon}^{(q-1)}(x, y, z) + E_{3,\epsilon}^{(q-2)}(x, y, z) + \dots,$$

$$E_{3,\epsilon}^{(q-1)}(x, y, z) = z^3 \left(\frac{\partial \alpha_{2,\epsilon}}{\partial y} - \frac{\partial \beta_{2,\epsilon}}{\partial x} \right).$$

Therefore

$$(5.6) \quad \frac{\partial \alpha_{2,\epsilon}}{\partial y} - \frac{\partial \beta_{2,\epsilon}}{\partial x} = 0.$$

The relations (5.5) and (5.6) imply that there exists a change of coordinates of the form (5.3) which reduces (5.1) to the form (5.4), where the functions $A_\epsilon^{(q)}$ and $B_\epsilon^{(q)}$ are affine with respect to \tilde{z} .

Our next step is a change of coordinates of the form

$$(5.7) \quad \hat{x} = \tilde{x} + r_\epsilon^{(q+2)}(\tilde{x}, \tilde{y}, \tilde{z}), \quad \hat{y} = \tilde{y}, \quad \hat{z} = \tilde{z},$$

where

$$\frac{\partial r_\epsilon^{(q+2)}}{\partial x} = -\tilde{c}_\epsilon^{(q+1)}.$$

It is easy to see that the change (5.7) reduces (5.4) to the form

$$\{\hat{x}, \hat{y}\} = \hat{z} + \hat{c}_\epsilon^{(q+2)}(\hat{x}, \hat{y}, \hat{z}) + \hat{c}_\epsilon^{(q+3)}(\hat{x}, \hat{y}, \hat{z}) + \dots,$$

$$(5.8) \quad \{\hat{y}, \hat{z}\} = a_\epsilon^{(0)}(\hat{x}, \hat{y}, \hat{z}) + a_\epsilon^{(1)}(\hat{x}, \hat{y}, \hat{z}) + \dots + a_\epsilon^{(q-1)}(\hat{x}, \hat{y}, \hat{z}) + A_\epsilon^{(q)}(\hat{x}, \hat{y}, \hat{z}) + \dots,$$

$$\{\hat{z}, \hat{x}\} = b_\epsilon^{(0)}(\hat{x}, \hat{y}, \hat{z}) + b_\epsilon^{(1)}(\hat{x}, \hat{y}, \hat{z}) + \dots + b_\epsilon^{(q-1)}(\hat{x}, \hat{y}, \hat{z}) + B_\epsilon^{(q)}(\hat{x}, \hat{y}, \hat{z}) + \dots.$$

Finally, to reduce (5.8) to the requirtd normal form (5.2) it suffices to make a change of coordinates

$$X = \hat{x}, \quad Y = \hat{y}, \quad Z = \hat{z} + \hat{c}_\epsilon^{(q+2)}(\hat{x}, \hat{y}, \hat{z}). \quad \blacksquare$$

6. Appendix. Normal form for integrable 1-forms

Let P be a local Poisson structure on \mathbb{R}^3 , and let Ω be a local nondegenerate volume form. Consider P as a field of 2-vectors; then we can associate to P a differential 1-form ω such that $\omega(Y) = \Omega(Y \wedge P)$ for any vector field Y . The relation $[P, P] = 0$, valid for any Poisson structure, implies the integrability of ω : $\omega \wedge d\omega = 0$. The integrable 1-form ω depends on the choice of Ω , and it is clear that ω is invariantly related to P up to multiplication by a nonvanishing function. Denote by (ω) the Pfaffian equation generated by ω , i.e., a module of differential 1-forms over the ring of smooth functions generated by ω . This Pfaffian equation is invariantly related to the Poisson structure P , and we will denote it by $b(P)$. If, in local coordinates, P has the form

$$\{x, y\} = B(x, y, z), \quad \{y, z\} = C(x, y, z), \quad \{z, x\} = D(x, y, z),$$

then the Pfaffian equation $b(P)$ is generated by the 1-form

$$B(x, y, z)dz + C(x, y, z)dx + D(x, y, z)dy.$$

Note that any local integrable Pfaffian equation (ω) has the form $b(P)$ for some Poisson structure P (invariantly related to (ω) up to multiplication by a nonvanishing function), therefore all the results of this paper imply immediate corollaries on the classification of integrable Pfaffian equations. In particular, the corollary of our main result, Theorem 1.1, is as follows.

THEOREM 6.1: *Let ω_ϵ be a local family of integrable 1-forms on \mathbb{R}^3 such that $\omega_0(0) = 0$ and $j_0^1\omega_0$ is not equivalent to $a(xdy - ydx)$, $a \in \mathbb{R}$. Then the family of Pfaffian equations (ω_ϵ) is equivalent to a family of the form*

$$(6.1) \quad (zdz + U_\epsilon(x, y, z)dx + V_\epsilon(x, y, z)dy),$$

and there exists a family of formal changes of the coordinates x, y, z , centered at the point $x = y = z = 0$ for all ϵ , reducing (6.1) to a family of Pfaffian equations generated by 1-forms

$$(6.2) \quad zdz + d\hat{f}_\epsilon(x, y) + zd\hat{g}_\epsilon(x, y),$$

where the formal series \hat{f}_ϵ and \hat{g}_ϵ satisfy the relation $d\hat{f}_\epsilon \wedge d\hat{g}_\epsilon \equiv 0$.

The singularities of integrable 1-forms were studied by many authors. The V singularities of Poisson structures correspond to the case where $d\hat{g}_0(0) \neq 0$, i.e., to the integrable Pfaffian equation generated by a 1-form ω such that $d\omega(0) \neq 0$. In this case one can use the Darboux theorem on classification of closed nondegenerated 2-forms to show that ω is equivalent to a 1-form of the form $a(x, y)dx + b(x, y)dy$. Therefore the classification of V singularities of integrable Pfaffian equations reduces to the orbital classification of vector fields on the plane. This well known reduction (see [Ku]) is similar to the results of section 2.1.

The $sl(2), so(3), A^+$ and A^- singularities of Poisson structures correspond, respectively, to the cases where in the normal form (6.2) the formal series \hat{f}_0 is R -equivalent to $x^2 + y^2$, $x^2 - y^2$, $x^2 \pm y^m$, $-x^2 \pm y^m$, $m \geq 3$, and \hat{g}_0 has the form $\hat{g}_0 = \hat{\lambda} \circ \hat{f}_0$ for some formal series $\hat{\lambda}$. These singularities of integrable Pfaffian equations are algebraically isolated. The results of the papers [Mo] and [Ma] imply that they can be reduced to a formal normal form (dW) , where $W = z^2 + x^2 \pm y^2$ in the $so(3)$ and $sl(2)$ cases, and $W = z^2 \pm x^2 \pm y^m$ in the A case. These normal forms also follow from the works [Co1], [Co2], [We1] and our results in section 3.

Using the 1-1 correspondence between Poisson structures and integrable Pfaffian equations, we can also establish corollaries of our results concerning the N singularities of integrable Pfaffian equations (the case where \hat{g}_0 is R -equivalent to $x^2 \pm y^2$ and $j_0^2\hat{f}_0 = 0$) and bifurcations of integrable Pfaffian equations near A and N singularities. As far as we know, these results and Theorem 6.1 are new.

Theorem 6.1 can be generalized to the n -dimensional case. Such a generalization is analogous to a generalization of Theorem 1.1 to Nambu structures. The corresponding results and their corollaries will be published elsewhere.

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